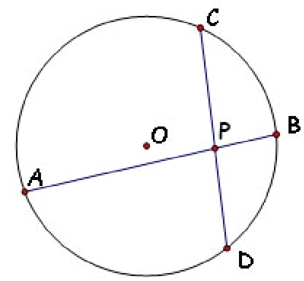
Math 5 – Trigonometry – Final Exam, Fall '08 Name___________________________

Show your work for credit. Write all responses on separate paper.

1. Show that if two chords of a circle, $\overline{AB}$ and $\overline{CD}$ intersect at P, then
   a. $\Delta APB \sim \Delta DPC$ are similar triangles. *Hint: You need to justify two congruent angles.*
   b. Use the proportionality of corresponding parts of similar triangles to show that the products are equal: $AP \cdot PB = DP \cdot PC$



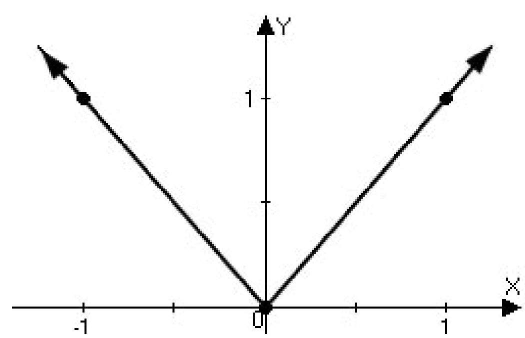
2. Find an equation for the line tangent to the circle $x^2 + y^2 = 1$ at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Recall that the tangent line is perpendicular to the radius line at the point of tangency.

3. Find an equation for the parabola with a vertex at (2, 3) and passing through (0, 0).

4. Find an equation for the sinusoid with amplitude = 1 and the largest period that will have peaks at (0,2) and (24, 2).

5. Consider the function $f(x) = \tan^{-1}(2x)$
   a. Find a formula for the inverse function, $y = f^{-1}(x)$
   b. What is the domain of $y = f^{-1}(x)$? Hint: as always, this is the same as the range of $y = f(x)$.
   c. Construct a large, careful graph of $y = f^{-1}(x)$ and $y = f(x)$ together, showing the symmetry through the line $y = x$.

6. Let $f(x) = |x|$, whose graph (shown at right) includes the points (-1, 1), (0,0) and (1,1).



   a. Write a formula for the function that results from transforming this function by stretching horizontally by a factor 2 and shifting up 1 and right 2. Make a table for and sketch a graph for this transformed function.
   b. Write a formula for the function that results from transforming this function by shifting down 2, reflecting across the x-axis and then shrinking vertically by a factor 1/2. Make a table and graph this transformed function.

7. A car's wheels have radius 30 cm and rolls at a constant speed so that the tires rotate 200 revolutions per minute. How far does the car travel in 10 minutes?

8. Consider the function $f(x) = 2\sin\left(\pi\left(x - \frac{1}{3}\right)\right)$

a. Find the amplitude, period and phase shift of the function.

b. Construct a large, careful graph of the function showing at least two periods of oscillation. Remember to scale and label your axes.

9. The point *P* is on the unit circle is in QIII and has $y = \sin(t) = \frac{-15}{17}$.
   a. Find the *x* coordinate of *P*.
   b. Find $\cos(t-\pi)$
   c. Find $\sin\left(t+\frac{\pi}{2}\right)$

10. Approximate the interior angles of the triangle with sides of length 13, 14, and 15 to the nearest ten thousandth of a radian.

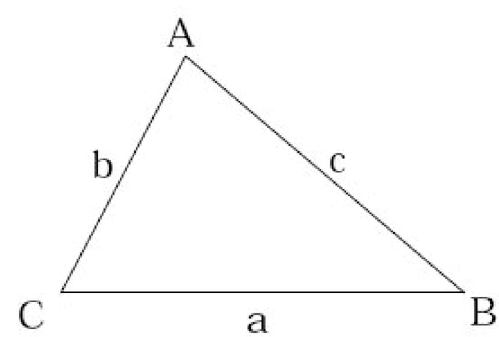
11. Construct a large, careful graph of the conic section showing all vertices, foci and asymptotes: $x^2 - 4y^2 = 1$

12. Write the conic in standard form and sketch a graph indicating key features:
   a. $x^2 + 4y^2 = 4x - y$

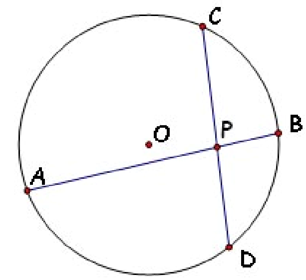
   b. $x = 4 + 2\sec t, \quad y = \tan t$

13. Consider the general triangle as shown at right.
   a. Use the formula for the area of a SAS defined triangle: $A = \frac{1}{2}xy\sin\theta$ to express the area of the triangle in three different ways.

   b. Set each of these expressions for the area equal to one another and thereby derive the law of sines.



Math 5 – Trigonometry – Final Exam Solutions Fall '08



1. Show that if two chords of a circle, $\overline{AB}$ and $\overline{CD}$ intersect at *P*, then
   c. $\Delta APC \sim \Delta DPB$ are similar triangles. *Hint: You need to justify two congruent angles.*
   SOLN: $\angle APC = \angle DPB$ are vertical angles and $\angle ACD = \angle ABD, \angle CDA = \angle ABC$ are pairs of inscribed angles subtended by the same arcs.
   d. Use the proportionality of corresponding parts of similar triangles to show that the products are equal: $AP \cdot PB = DP \cdot PC$
   SOLN: Since corresponding parts of similar triangles are proportional,
   $\frac{AP}{DP} = \frac{PC}{PB} \Rightarrow AP \cdot PB = DP \cdot PC$

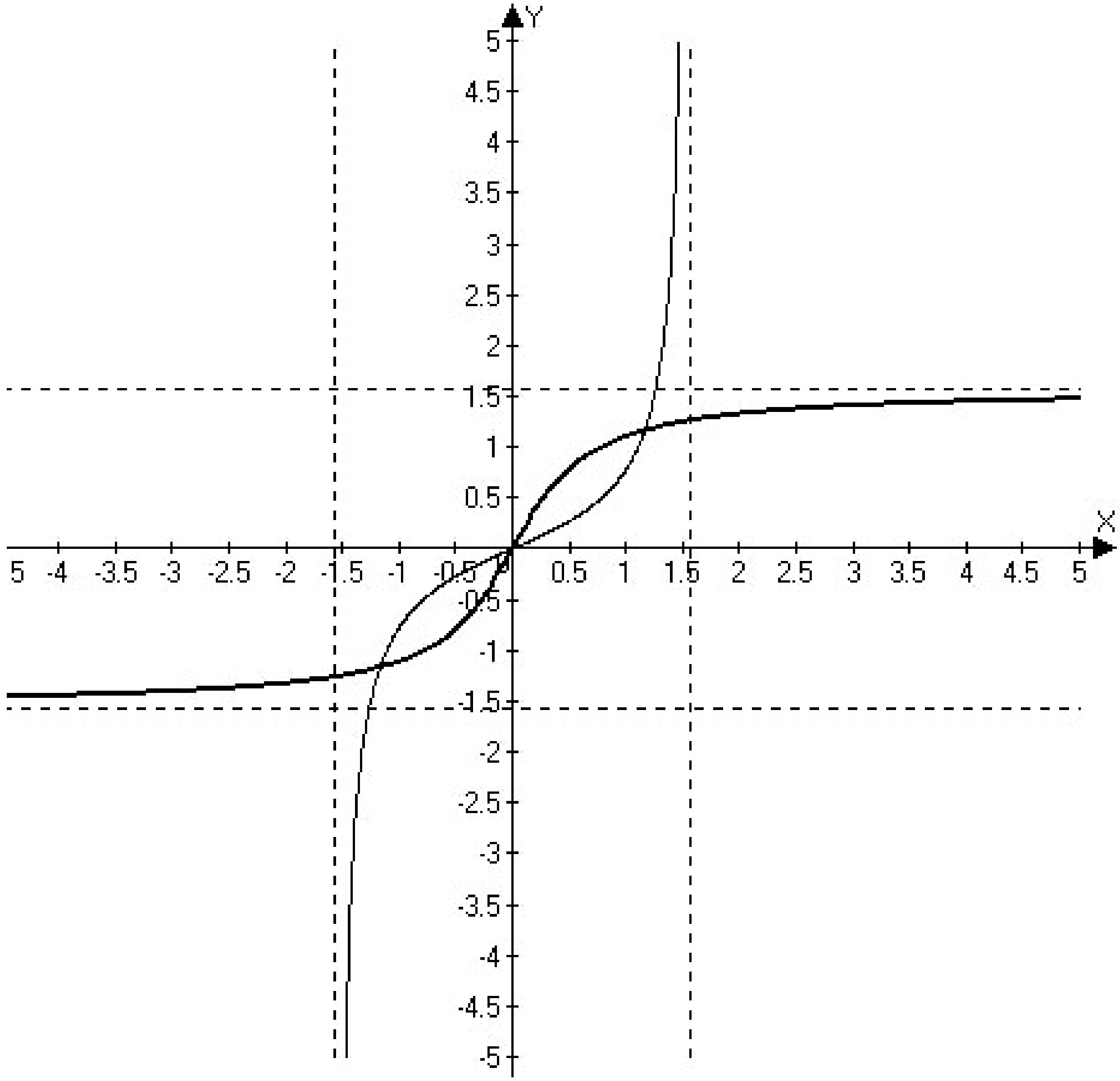
2. Find an equation for the line tangent to the circle $x^2 + y^2 = 1$ at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Recall that the tangent line is perpendicular to the radius line at the point of tangency.
   SOLN: The tangent line's slope is the negative reciprocal of the radius' slope: $\frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}$.
   Plugging into the point-slope equation, then: $y - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{3}\left(x - \frac{1}{2}\right) \Leftrightarrow y = -\frac{\sqrt{3}}{3}x + \frac{2\sqrt{3}}{3}$

3. Find an equation for the parabola with a vertex at (2, 3) and passing through (0, 0).
   SOLN: $y = a(x-2)^2 + 3 = -\frac{3}{4}(x-2)^2 + 3$

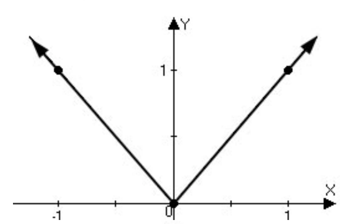
4. Find an equation for the sinusoid with amplitude = 1 and the largest period that will have peaks at (0,2) and (24, 2).
   SOLN: $y = 1 + \cos\left(\frac{\pi x}{12}\right)$

5. Consider the function $f(x) = \tan^{-1}(2x)$
   a. Find a formula for the inverse function, $y = f^{-1}(x)$
   SOLN: $f^{-1}(x) = \frac{1}{2}\tan(x)$
   b. What is the domain of $y = f^{-1}(x)$? Hint: as always, this is the same as the range of $y = f(x)$.
   SOLN: Domain = $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

c. Construct a large, careful graph of $y = f^{-1}(x)$ and $y = f(x)$ together, showing the symmetry through the line $y = x$.

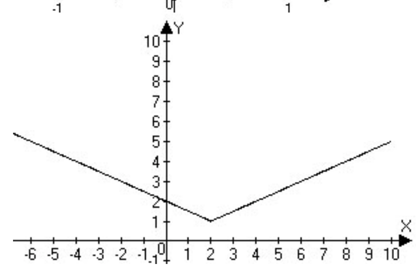


6. Let $f(x) = |x|$, whose graph (shown at right) includes the points (-1, 1), (0,0) and (1,1).



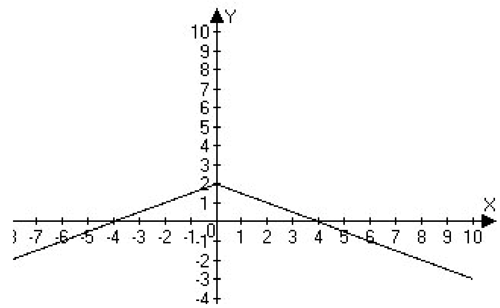
a. Write a formula for the function that results from transforming this function by stretching horizontally by a factor 2 and shifting up 1 and right 2. Make a table for and sketch a graph for this transformed function.

SOLN: $f(x) = \frac{1}{2}|x-2|+1$



b. Write a formula for the function that results from transforming this function by shifting down 2, reflecting across the x-axis and then shrinking vertically by a factor 1/2. Make a table and graph this transformed function.

SOLN: $-\frac{1}{2}f(x)+2=2-\frac{1}{2}|x|$



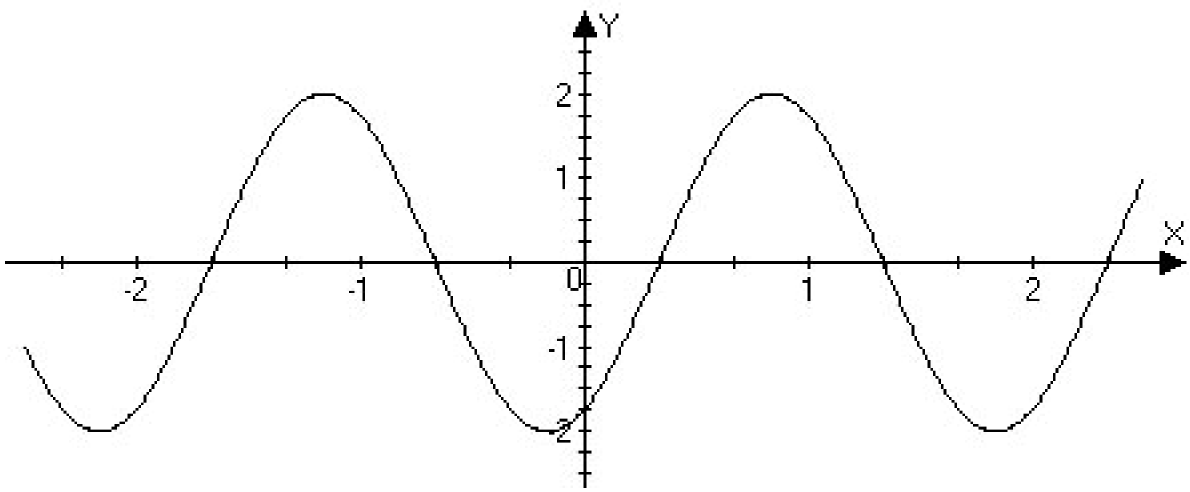
7. A car's wheels have radius 30 cm and rolls at a constant speed so that the tires rotate 200 revolutions per minute. How far does the car travel in 10 minutes?

   SOLN: $200\frac{\text{rev}}{\text{min}}\times\frac{2\pi(30\text{cm})}{\text{rev}}=12000\pi\frac{\text{cm}}{\text{min}}\times 10\,\text{min}=120000\pi\text{cm}=1.2\pi\text{km}\approx 3.77\text{km}$

8. Consider the function $f(x)=2\sin\left(\pi\left(x-\frac{1}{3}\right)\right)$

   a. Find the amplitude, period and phase shift of the function.
   SOLN:Amplitude = 2; period = 2 and phase shift = 1/3

   b. Construct a large, careful graph of the function showing at least two periods of oscillation. Remember to scale and label your axes.



9. The point P is on the unit circle is in QIII and has $y=\sin(t)=\frac{-15}{17}$.

   a. Find the x coordinate of P.

   SOLN: Recall the Pythagorean triple, 8-15-17, so $x=-\frac{8}{17}$

   b. Find $\cos(t-\pi)$

   SOLN: This is the opposite of cos(t). 8/17

   c. Find $\sin\left(t+\frac{\pi}{2}\right)$

   SOLN: This is the y coordinate of a quarter turn counter clockwise from P: -8/17

10. Approximate the interior angles of the triangle with sides of length 13, 14, and 15 to the nearest ten thousandth of a radian.

SOLN: By the law of cosines,

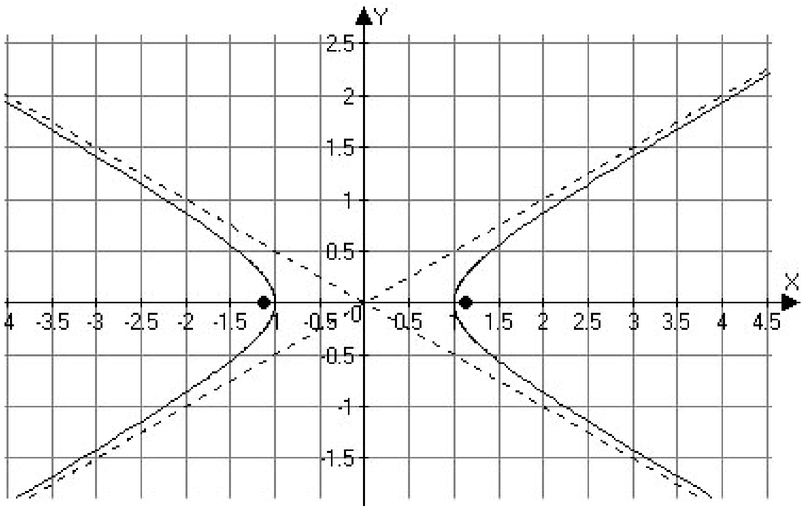
$$15^2 = 13^2 + 14^2 - 2(13)(14)\cos\theta \Leftrightarrow \theta = \cos^{-1}\left(\frac{169+196-225}{364}\right) = \cos^{-1}\left(\frac{5}{13}\right) \approx 1.176$$

By the law of sines, $\frac{\sin\theta}{14} \approx \frac{\sin(1.176)}{15} \Leftrightarrow \theta \approx \sin^{-1}\left(\frac{14\sin(1.176)}{15}\right) \approx 1.038$

Thus the angle opposite 14 is approximately 3.142 – 1.176 – 1.038 is about 0.927

11. Construct a large, careful graph of the conic section showing all vertices, foci and asymptotes: $x^2 - 4y^2 = 1$

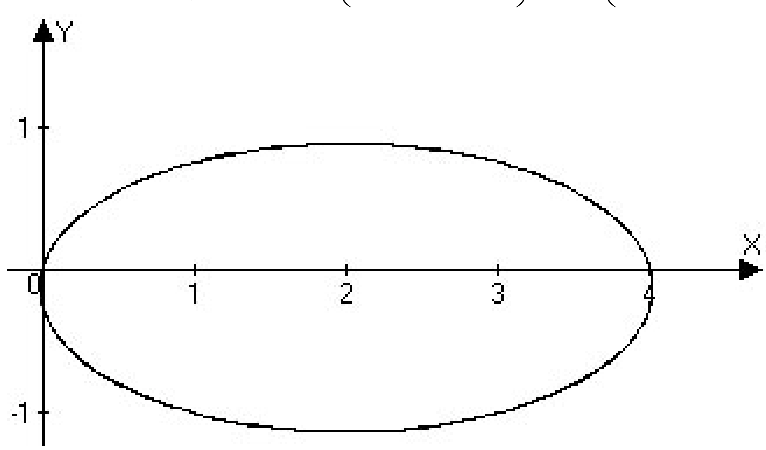
SOLN: The vertices are at (1,0) and (–1 ,0) and the foci are at $\left(\pm\sqrt{5}/2, 0\right)$



12. Write the conic in standard form and sketch a graph indicating key features:

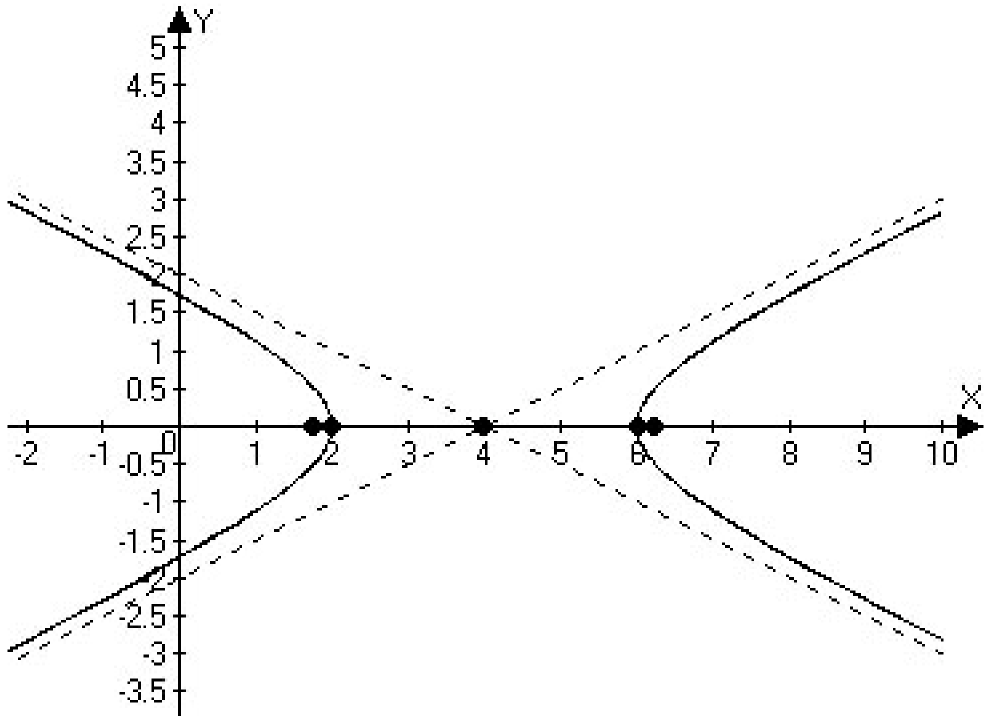
a. $x^2 + 4y^2 = 4x - y \Leftrightarrow (x-2)^2 + 4\left(y+\frac{1}{8}\right)^2 = 4 + \frac{1}{16} \Leftrightarrow \frac{(x-2)^2}{65/16} + \frac{\left(y+\frac{1}{8}\right)^2}{65/64} = 1$

Center $\left(2, -\frac{1}{8}\right)$, vertices $\left(2 \pm \frac{\sqrt{65}}{4}, -\frac{1}{8}\right)$ and $\left(2, \frac{-1}{8} \pm \frac{\sqrt{65}}{8}\right)$ & foci at $\left(2 \pm \frac{\sqrt{195}}{8}, -\frac{1}{8}\right)$

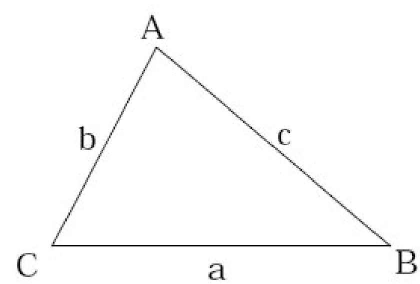


b. $x = 4 + 2\sec t, \quad y = \tan t$

Center (4,0), vertices (2,0),(6,0), foci $\left(\pm\sqrt{5},0\right)$ and asymptotes $y =$ $y = \pm\left(x-4\right)/2$



13. Consider the general triangle as shown at right.



a. Use the formula for the area of a SAS defined triangle: $A = \frac{1}{2}xy\sin\theta$ to express the area of the triangle in three different ways.

SOLN: $A = \frac{1}{2}ab\sin\angle C = \frac{1}{2}ac\sin\angle B = \frac{1}{2}bc\sin\angle C$

b. Set each of these expressions for the area equal to one another and thereby derive the law of sines. SOLN: multiply through by 2/(*abc*).